7.2 Angle Relationships in Quadrilaterals

Many quadrilaterals are visible in this photograph of the recent addition to the Ontario College of Art and Design. Quadrilaterals are important shapes in design and construction. Quadrilateral shapes appear in both ordinary and unusual new buildings, as well as in everyday objects all around you.

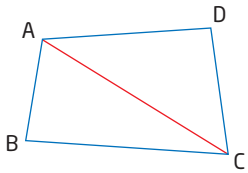
Investigate

Tools
- ruler
- protractor

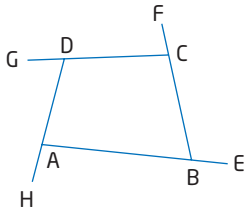
How are the interior and exterior angles of a quadrilateral related?

Method 1: Use Pencil and Paper

1. Draw a large quadrilateral. Label the vertices.
2. Use a protractor to measure each of the four interior angles.
3. Find the sum of the four interior angles.
4. Compare your results with those of your classmates.
5. **Reflect** Make a hypothesis about the sum of the interior angles of any quadrilateral. Describe how you can test your hypothesis.
6. Any quadrilateral can be divided into two triangles by constructing a diagonal like line segment AC. How are the angles in the two triangles related to the interior angles of the quadrilateral?



7. **Reflect** Explain how you can show that the sum of the interior angles is the same for all quadrilaterals.
8. Extend one side at each vertex of your quadrilateral to create an exterior angle. Name and measure the four exterior angles. Find the sum of these exterior angles.
9. **Reflect** Make a hypothesis about the sum of the exterior angles of any quadrilateral. Describe how you can test your hypothesis.



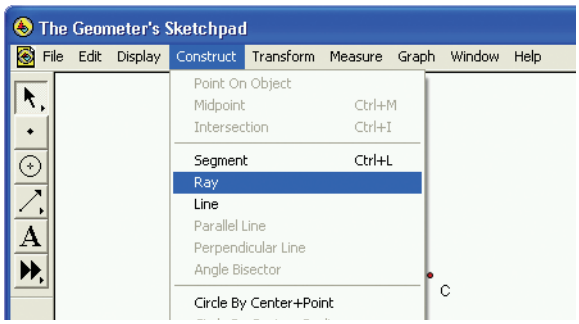
Method 2: Use *The Geometer's Sketchpad®*

Tools
- computer with *The Geometer's Sketchpad®*

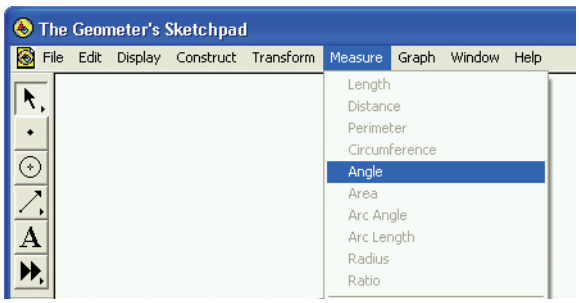
1. Turn on automatic labelling of points. From the **Edit** menu, choose **Preferences**, click on the **Text** tab, check **For All New Points**, and click on **OK**.

2. Use the **Point Tool** to create four points on the screen.

3. Select point A and then point B. From the **Construct** menu, choose **Ray**. Use the same method to construct rays from B to C, from C to D, and from D to A.



4. To measure ∠DAB, select points D, A, and B in that order. From the **Measure** menu, choose **Angle**. Use the same method to measure the other three interior angles of the quadrilateral.



5. From the **Measure** menu, choose **Calculate**. Click on the measure for ∠DAB. Click + on the calculator; then, click on the measure for ∠ABC. Add the other two interior angles to the calculation.

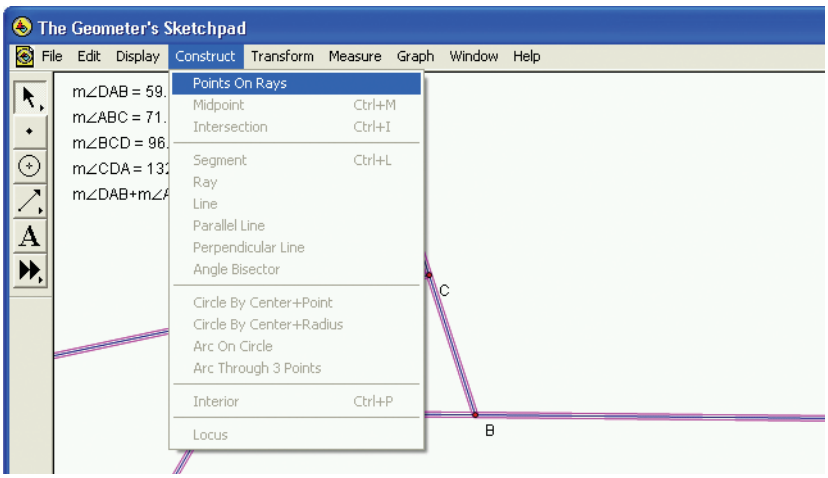
6. Make a hypothesis about the sum of the interior angles of any quadrilateral.

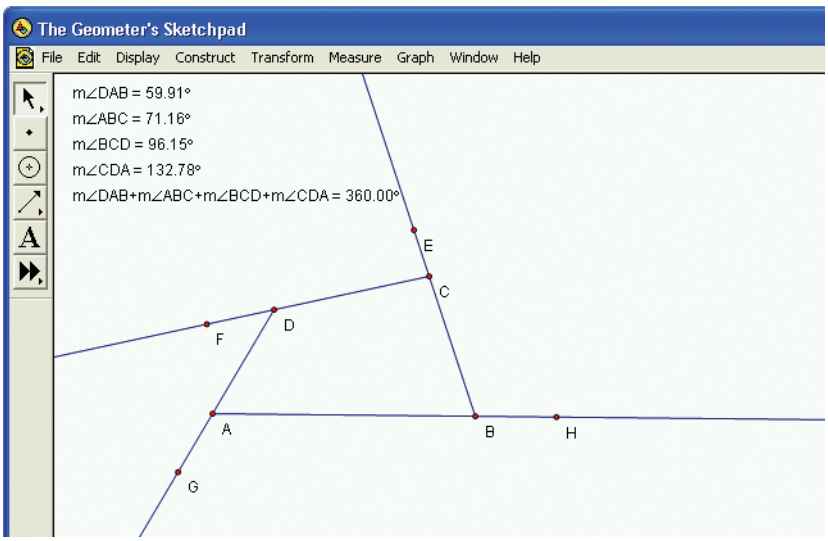
7. Drag one of the vertices around the screen. Watch the sum of the interior angles as you move the vertex. Try moving the other vertices as well.

8. **Reflect** What can you conclude about the sum of the interior angles of any quadrilateral? Do your observations support your hypothesis? Explain.

9. Make a hypothesis about the sum of the exterior angles in any quadrilateral.

10. Select the four rays. From the **Construct** menu, choose **Points on Rays**. If necessary, drag each point to a location outside the quadrilateral.





11. Use the Measure and Calculate tools to test your hypothesis about the sum of the exterior angles.

12. **Reflect** Is your hypothesis correct? Explain.

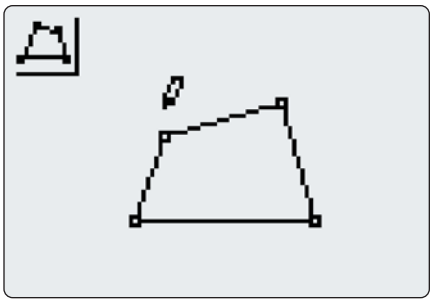
Method 3: Use a Graphing Calculator

Tools

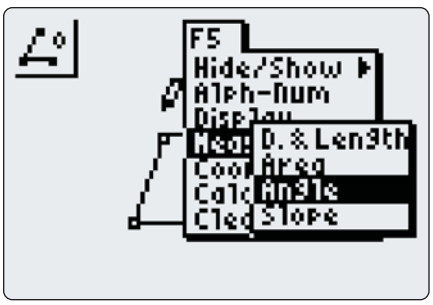
- TI-83 Plus or TI-84 graphing calculator

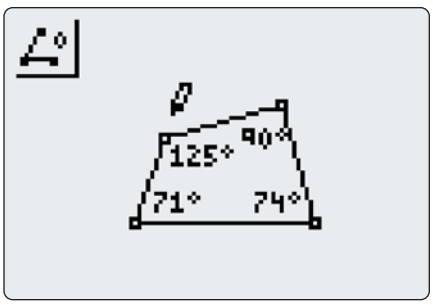
1. Press APPS and select **CabriJr**. Press ENTER when the title screen appears. If you need to clear a previous drawing from the screen, press Y= to display the **F1** menu, and select **New**.

2. Press WINDOW to display the **F2** menu. Select **Quad.**. Use the cursor keys to move the pencil cursor around the screen. Press ENTER to place each vertex of the quadrilateral.

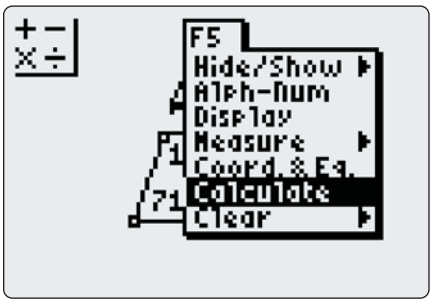


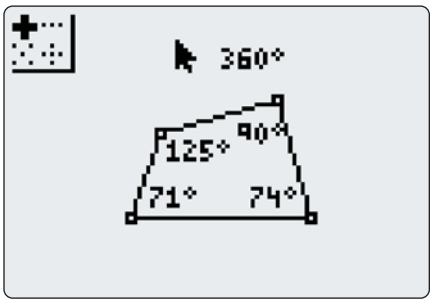
3. To measure angles, press GRAPH to display the **F5** menu, highlight **Measure**, and press ▶. Select **Angle** from the submenu. For each angle, select three points that define the angle by pressing ENTER at each point. Always select the vertex as the second of the three points. Use the cursor keys to drag the measurement so that it labels the angle clearly; then, press ENTER. Measure the four interior angles of the quadrilateral.





4. Press GRAPH to display the **F5** menu, and select **Calculate**. Select three of the angle measures by moving the cursor to each one and pressing ENTER when the measurement is underlined. Press + and drag the subtotal to an empty part of the screen. Select this subtotal and the fourth angle measure. Now, press + to display the sum of all four angles. To avoid confusion, hide the subtotal by dragging the total sum to the same location.





Technology Tip

Version 2.00 of Cabri® Jr. can add only three angles at a time. Later versions may let you add all four angles of the quadrilateral in a single operation.

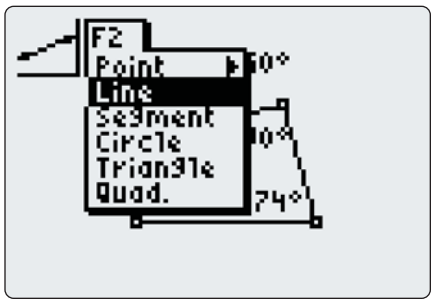
5. **Reflect** Make a hypothesis about the sum of the interior angles of any quadrilateral.

6. Press CLEAR. Move the cursor to one of the vertices and press ALPHA. Now, use the cursor keys to drag the vertex to various new locations. Watch the sum of the interior angles as you move the vertex. Try moving the other vertices as well.

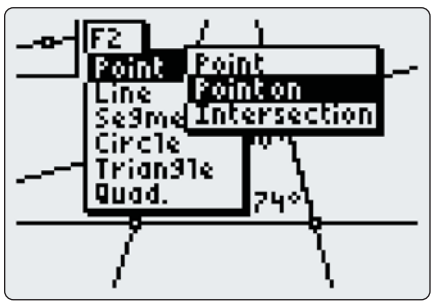
7. **Reflect** What can you conclude about the sum of the interior angles of any quadrilateral? Do your observations support your hypothesis? Explain.

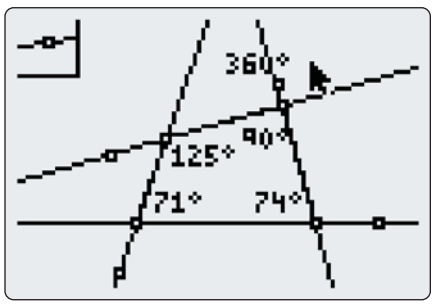
8. Make a hypothesis about the sum of the exterior angles for any quadrilateral.

9. Press WINDOW to display the **F2** menu and select **Line**. Move the cursor to a point on one side of the quadrilateral and press ENTER. Then, move the cursor to another point on the side and press ENTER again. Use the same method to extend the other three sides.



10. Press WINDOW to display the **F2** menu. With the cursor on **Point**, press ▶. Select **Point on** from the submenu. To place a point for an exterior angle, move the cursor onto the portion of a line outside the quadrilateral and press ENTER. Place similar points on the other three lines.





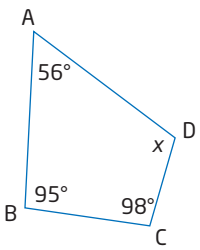
11. Use the **Measure** and **Calculate** functions to find the sum of the exterior angles. Then, test whether moving the vertices affects this sum.

12. **Reflect** What can you conclude about the sum of the exterior angles of any quadrilateral? Do your observations support your hypothesis? Explain.

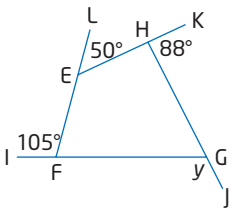
Example 1 Measures of Angles of a Quadrilateral

Find the measure of the unknown angle in each quadrilateral.

a)



b)



Solution

a) Since the sum of the interior angles of a quadrilateral is 360°,

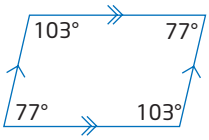
$$\angle A + \angle B + \angle C + \angle D = 360°$$
$$56° + 95° + 98° + x = 360°$$
$$249° + x = 360°$$
$$x = 360° - 249°$$
$$x = 111°$$

b) Since the sum of the exterior angles of a quadrilateral is 360°,

$$\angle EFI + \angle LEH + \angle KHG + \angle JGF = 360°$$
$$105° + 50° + 88° + y = 360°$$
$$243° + y = 360°$$
$$y = 360° - 243°$$
$$y = 117°$$

Example 2 Angle Relationships in Parallelograms

Use this diagram to determine two of the angle properties of a parallelogram.



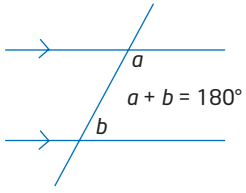
Solution

Examining the measures of the interior angles of this parallelogram shows that

- **adjacent** angles are **supplementary**
- angles at opposite vertices are equal

You can show that these relationships apply to all parallelograms.

Where a **transversal** crosses two parallel lines, angles that form a "C" pattern are supplementary.

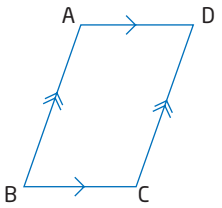


adjacent
- adjoining or next to

supplementary
- adding to 180°

transversal
- line intersecting two or more lines

In any parallelogram ABCD, AD is a transversal between the parallel sides AB and CD. So,



$\angle CDA + \angle DAB = 180°$

Applying the same reasoning to each side gives

$\angle DAB + \angle ABC = 180°$
$\angle ABC + \angle BCD = 180°$
$\angle BCD + \angle CDA = 180°$

Thus, adjacent angles in a parallelogram are supplementary.

This angle property leads to a relationship between opposite angles.

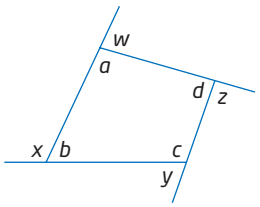
Since $\angle CDA + \angle DAB = 180°$, $\angle CDA = 180° - \angle DAB$.
Since $\angle DAB + \angle ABC = 180°$, $\angle ABC = 180° - \angle DAB$.
Therefore, $\angle CDA = \angle ABC$.

In the same way, you can show that $\angle BCD = \angle DAB$.

Therefore, opposite angles in a parallelogram are equal.

Key Concepts

- The sum of the interior angles of a quadrilateral is 360°.
- The sum of the exterior angles of a quadrilateral is 360°.

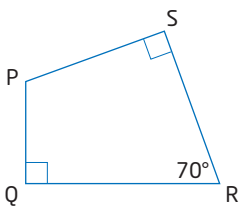


Interior angles:
$a + b + c + d = 360°$

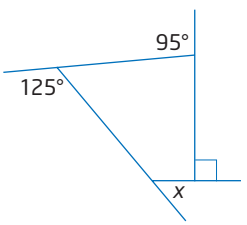
Exterior angles:
$w + x + y + z = 360°$

Communicate Your Understanding

C1 Calculate the measure of $\angle P$. Explain your calculation.



C2 Omar calculates that x represents an angle measure of 50°. Is he correct? How do you know?

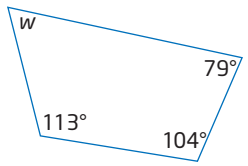


Practise

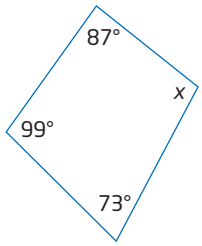
For help with questions 1 to 4, see Example 1.

1. Find the angle measures w, x, y, and z.

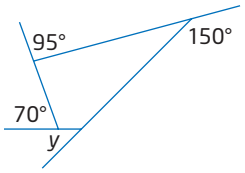
a)



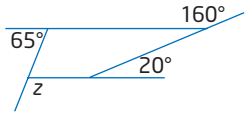
b)



c)



d)



2. The measures of three of the interior angles of a quadrilateral are 40°, 90°, and 120°. The measure of the fourth interior angle is

A 110° **B** 130° **C** 210° **D** 250°

3. The measures of exterior angles at three vertices of a quadrilateral are 80°, 100°, and 120°. The measure of an exterior angle at the fourth vertex is

A 40° **B** 60° **C** 100° **D** 140°

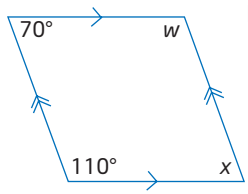
4. Each row of this table lists measures of three interior angles in a quadrilateral. Find the measure of the fourth interior angle in each quadrilateral.

	∠A	∠B	∠C	∠D
a)	100°	75°	50°	unknown
b)	20°	35°	unknown	150°
c)	70°	unknown	70°	70°
d)	unknown	90°	90°	90°

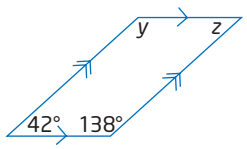
For help with question 5, see Example 2.

5. Find the measure of each unknown angle.

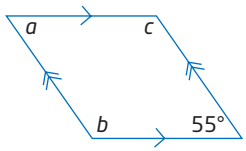
a)



b)



c)

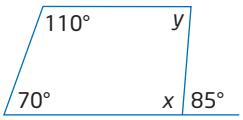


6. What angle property do triangles and quadrilaterals have in common?

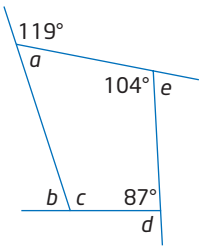
Connect and Apply

7. Find the measure of each unknown angle.

a)



b)



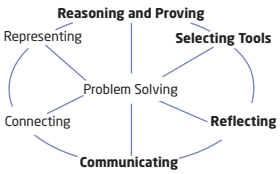
8. What is the minimum number of angles you need to measure to calculate the measure of all of the interior and exterior angles of a quadrilateral? Justify your answer.

9. Draw a quadrilateral with each set of interior angles, or explain why the quadrilateral is not possible. Calculate the measure of the fourth angle where possible.

a) $\angle A = 170°$, $\angle B = 65°$, and $\angle C = 160°$

b) $\angle E = 60°$, $\angle F = 75°$, and $\angle G = 120°$

c) $\angle P = 30°$, $\angle Q = 65°$, and $\angle R = 60°$



10. Draw an example of a quadrilateral with each set of interior angles, or explain why the quadrilateral is not possible.

a) four obtuse angles

b) exactly two obtuse angles

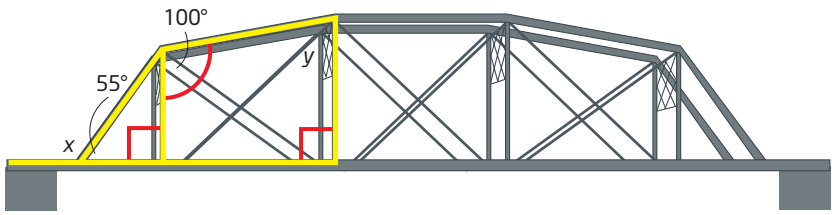
c) one obtuse angle and three acute angles

d) one obtuse angle and two right angles

e) exactly three right angles

11. Calculate the mean measure for the interior angles of a quadrilateral.

12. This diagram shows the structure of a bridge over the river between Ottawa and Gatineau.



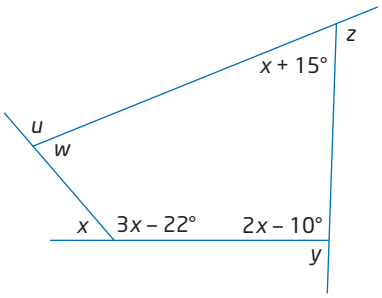
a) Calculate the measure of the exterior angle between the road and the foot of the triangle at the left end of the bridge.

b) Calculate the angle at the upper right corner of the quadrilateral on the left side of the bridge.

c) Why did the bridge's designers use these shapes?

Extend

13. Find the measure of each unknown angle.



14. Draw a square ABCD and a rectangle PQRS. Construct diagonals AC and PR. Investigate the properties of the two quadrilaterals to answer parts a) to c). Describe how you determined the answer to each question.

a) Does each diagonal divide the quadrilateral into two **congruent** triangles?

b) Is the diagonal a line of symmetry in either of the quadrilaterals?

c) Does the diagonal bisect any angles in the quadrilateral?

congruent

- equal in all respects

15. a) Draw a quadrilateral, and place a point E anywhere inside it. Then, draw a line segment from E to each vertex.

b) Find the sum of the four angles at point E.

c) Find the sum of all of the interior angles of the four triangles inside the quadrilateral.

d) Use your diagram to show that the sum of the interior angles of a quadrilateral is 360°.

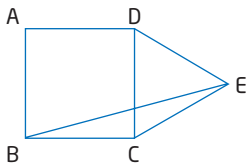
16. Find the measures of the interior angles of a quadrilateral so that the measures have each ratio. Then, sketch each quadrilateral.

a) 1:1:1:1 **b)** 1:1:2:2 **c)** 1:2:3:4 **d)** 3:4:5:6

17. Quadrilaterals that can be inscribed in a circle with all four vertices on the circumference of the circle are called *cyclic quadrilaterals*. Investigate angle relationships in cyclic quadrilaterals. Write a brief report of your findings.

18. Math Contest ABCD is a square and △DCE is equilateral. The measure of ∠CEB is

A 10° **B** 15° **C** 20° **D** 30°



19. Math Contest Given four sides with unequal lengths, how many non-congruent quadrilaterals can you make?